Conservation laws of differential equations beyond Lagragian methods: Exponential symmetries

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Resumen

We have derived first integrals of ordinary differential equations via exponential symmetries. We have constructed a formal Lagrangian and the adjoint system. From the adjoint system we have derived a first integral depending on the nonlocal variable. After obtain two first integrals of the nonlocal variable we were able to derive proper first integrals eliminating the nonlocal variable. We have applied this approach to the second order and third order Riccati equations. In order to do this study, three different programs have been designed with free software MAXIMA. These programs allow us to determine the adjoint equation, the conditions of self-adjointness and the first integrals of the Ricatti equations.

Introduction

One of the important applications of Lie group theory is to use the Lie symmetry group to construct conservation laws of the differential equations. This can be done if a Lagrangian exists and the symmetries are variational ones. Although Noether's approach provides an elegant algorithm for finding conservation laws, it possesses a strong limitation: it can only be applied to equations having variational structure.

During the project we have developed a scientific paper with the theoretical results that we have obtained and the application of these results to Riccati III equation. For the sake of completeness it will be interesting to apply these results to different equations.

Project

1. Riccati III

In [2] we studied nonlocal symmetries of Riccati chains and studied similarity reductions coming out from Riccati III equation

$$F = u_{xxx} + 4uu_{xx} + 3u_x^2 + +6u^2u_x + u^4 = 0 \tag{1}$$

was treated as system

$$\begin{array}{l} u_{xxx} + 4uu_{xx} + 3u_x^2 + 6u^2u_x + u^4 = 0 \\ v_x = f(u, v). \end{array} \tag{2}$$

Our studies showed that the entire Riccati chain of equations admit the same form of nonlocal symmetry. As a consequence the similarity variables are also found to be same for all the equations in the chain. The similarity reduced N^{th} order ODE, $N = 2, 3, 4, \cdots$, ends at $(N-1)^{th}$ order ODE in the Riccati chain. From the solution of the $(N-1)^{th}$ order ODE we derive the general solution for the N^{th} order ODE.

In [2] we constructed nonlocal symmetries of Eq. (1) we consider the system of equations (2). We found that the infinitesimal generator

$$\mathbf{v} = c(x)e^v u\frac{\partial}{\partial u} + c(x)e^v\frac{\partial}{\partial v}, \tag{3}$$

with

$$f(x, u) = -u - \frac{c_x}{c},$$

again be a solution for these determining equations as well. As a consequence the characteristic equation provides the same form of similarity variables

$$z = x,\ \zeta = \frac{u_x}{u} + u. \tag{4}$$

for the Eq.(1). In this project we have obtained the following results

Theorem 1. Given equations (2), the adjoint equations to system (2) are

$$F_1^* \equiv \ v_1 - u_{1xxx} + 4uu_{1xx} + 2u_xu_{1x} - 6u^2u_{1x} + 2u_{xx}u_1 + 4u^3u_1, \tag{5}$$

$$F_2^* \equiv \ -v_{1x}. \tag{6}$$

In order to solve system (5) we consider $u_1 = h(x, t, u, u_x, u_{xx})$ and $v_1 = g(x, t, u, u_x, u_{xx})$. Substituting u_1 and v_1 into (6) we obtain

$$-g_{u_{xx}}\, u_{xxx} - g_{u_x}\, u_{xx} - g_v\, v_x - g_u\, u_x - g_x = 0.$$

From this equation we deduce that $g = k$ is constant. Substituting g, u_1 and v_1 into (5) we obtain a very long equation. It is impossible to solve this equation in general, but by using an ansatz we obtain

$$h_1 = -\frac{k_1\,(3\,u_x+u^2)\,x}{(u_{xx}+3\,u\,u_x+u^3)^3},\quad h_2 = \frac{-u_x\,P(x)-u^2\,P(x)+u\,P'(x)-\frac{1}{2}P''(x)}{(u_{x\,x}+3\,u\,u_x+u^3)^2},$$

with $P(x) = k_4\,x^4 + k_3\,x^3 + k_2\,x^2 + k_1\,x + k_0$

We consider the generator (3). Substituting the infinitesimals we have

$$W_1 = c\,u\,e^v, \qquad W_2 = c\,e^v. \tag{7}$$

First we substitute the normal vector (7) and the lagrangian

$$\mathcal{L} = v_x\,v_1 + u\,v_1 + \frac{c_x\,v_1}{c} + u_{x\,x\,x}\,u_1 + 4\,u\,u_{x\,x}\,u_1 + 3\,(u_x)^2\,u_1 + 6\,u^2\,u_x\,u_1 + u^4\,u_1$$

by making the change de φ we get the first integral

$$\begin{aligned} FI \equiv e^v \Big(& cv_1 + cuu_1v_{xx} + cuu_1\,(v_x)^2 - cuu_{1x}v_x + 2cu_xu_1v_x + 4cu^2u_1v_x + 2c_xuu_1v_x + cuu_{1xx} - cu_xu_{1x} \\ & -4cu^2u_{1x} - c_xuu_{1x} + cu_{xx}u_1 + 6cuu_xu_1 + 2c_xu_xu_1 + 6cu^3u_1 + 4c_xu^2u_1 + c_{xx}uu_1 \Big). \end{aligned}$$

From first equation of (2) we obtain u_{xxx}, and from second equation of (2) we obtain v_x. Substituting u_{xxx} and v_x into (8) we get the first integral. For $u_1 = h$ and $v_1 = 0$ given in (**??**) we obtain

$$\begin{aligned} FI \equiv -\frac{ce^v}{(u_{xx}+3uu_x+u^3)^2} \Big(& 6k_4u_{xx}x^2 + 18k_4uu_xx^2 + 6k_4u^3x^2 + 3k_3u_{xx}x + 9k_3uu_xx - 12k_4u_xx \\ & +3k_3u^3x - 12k_4u^2x + k_2u_{xx} + 3k_2uu_x - 3k_3u_x + k_2u^3 - 3k_3u^2 + 12k_4u \Big) \end{aligned}$$

2. Activity carried out

We made a stay in the Department of Finance and Management Science, Norwegian School of Economics. During this stay we have been working with professor Kozlov and with have talks and seminars with other professors of the Department.

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Referencias

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